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NP-completeness of optimal planning problem for modular robots

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Abstract

Self-reconfigurable modular robots (SRM-robots) can autonomously change their shape according to different tasks and work environments, and have received considerable attention recently. Many reshaping/reconfiguration algorithms have been proposed. In this paper, we present a theoretical analysis of computational complexity on a reshape planning for a kind of lattice-type 3D SRM-robots, whose modules are of cubic shape and can move by rotating on the surfaces of other modules. Different from previous NP-completeness study on general chain-type robots (i.e. the motion of any chains and the location of modules can be arbitrary), we consider more practical constraints on modules' shape (i.e. cubic shape), position (lying in 2D/3D grids) and motion (using orthogonal rotations) in this paper. We formulate the reshape planning problem of SRM-robots with these practical constraints by a (p, q) optimization problem, where p and q characterize two widely used metrics, i.e. the number of disconnecting/reconnecting operations and the number of reshaping steps. Proofs are presented, showing that this optimization problem is NP-complete. Therefore, instead of finding global optimization results, most likely approximation solution can be obtained for the problem instead of seeking polynomial algorithm. We also present the upper and lower bounds for the 2-tuple (p, q), which is useful for evaluating the approximation algorithms in future research.

Keywords NP-completeness · Reconfigurable modular robots · Optimization problem

1 Introduction

A modular robot consists of a number of mechatronic modules, each is physically independent and encapsulates a certain simple function. Complex tasks can be realized by the joint function of modules on such robots. Self-reconfigurable modular robots (SRM-robots) can autonomously change their shape according to different tasks or different work environments, and thus, have attracted a lot of attention in the last decade (Ahmadzadeh and Masehian 2015; Liu et al. 2018; Stoy et al. 2010).

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The *shape* of a SRM-robot can be defined by the positions of its constituent modules (Stoy and Brandt 2013). A *configuration* is a shape with additional consideration including orientations, connectors and possibly different gender types. Both shape and configuration are important aspects and received considerable attention: different shapes—among which a SRM-robot can transform—can help improve the human spatial ability (Yu et al. 2019a, b), while different configurations can endow SRM-robots with different locomotion capacities (Stoy et al. 2010). In this paper, we focus on the shape of SRM-robots and we call the process of transforming a SRM-robot from an initial shape into a target shape as *self-reshaping*. In the reshaping process, all modules in a SRM-robot are always connected/touched and an elementary operation consists of three steps: *disconnecting* one or more modules, *moving* and *re-connecting* these modules. In this paper, we study the computational complexity of the optimal *reshaping* planning problem for SRM-robots.

Many types of modular robots exist. In our study, we focus on an important class of 3D SRM-robots, whose modules are of cubic shape and can move by rotating on the surfaces of other modules. Moreover, to place the rotating modules on other modules' surface, orthogonal rotation is applied, i.e.

(a) M-TRAN **(b)** SuperBot **(c)** SMORES **(d)** M-Block

Fig. 1 All these robot modules are of cubic shape and can move by rotating on the surfaces of other modules. These pictures are courtesy of Prof. Haruhisa Kurokawa (M-TRAN), Prof. Wei-Min Shen (SuperBot), Prof. Mark Yim (SMORES) and Prof. John William Romanishin (M-Block)

the angle in each rotation operation is $\frac{\pi}{2} \cdot I$, where I is a non-negative integer. We call this class of modules *rotatable cubic (RC) modules*. Some well known SRM-robots in this class include M-TRAN series (Kurokawa et al. 2003, 2008; Murata et al. 2002), SuperBot (Salemi et al. 2006), EasySRRobot (Yu et al. 2017), SMORES (Davey et al. 2012; Jing et al. 2017) and M-Block (Romanishin et al. 2013, 2015); see Fig. 1 for some examples.

Many reshaping/reconfiguration planning methods applied for RC modules have been proposed (Asadpour et al. 2008; Jing et al. 2017; Pamecha et al. 1997; Sung et al. 2015; Yu et al. 2019b). However, these methods only provide feasible solutions and do not consider the optimal solution, e.g. achieving the least number of reshape/reconfiguration steps. This optimization problem is difficult, since given n modules, the possible shapes and configurations of a SRM-robot are exponential in n (Chirikjian et al. 1996; Stoy and Brandt 2013). Hou and Shen (2010, 2014) presented an elegant complexity analysis on optimal reconfiguration planning problem for chain-type modular robots. They show that the optimal reconfiguration problem is NP-complete and then a polynomial algorithm for this problem is unlikely to exist. Another NP-completeness proof for the same problem was later presented in Gorbenko and Popv (2012). Given that this general optimization problem is NP-complete, Hou and Shen (2014) further proposed two novel heuristic strategies—MDCOP and GreedyCM—to make a good tradeoff between planning optimization and running time. However, these existing NP-completeness analyses are all based on general *chain-type* robots: this kind of robots consist of chains of modules and the robots can freely move any one of these chains in any positions. As a comparison, in this paper we study SRM-robots consisting of RC modules with more practical constraints:

- Each RC module is of cubic shape and moved by rotation.
- In each rotation operation, two or more connected RC modules can be moved. This consideration is based on the fact that the actuators in most RC modules have enough torque to drive multiple modules; e.g. a M-TRAN II (Kurokawa et al. 2003) or M-TRAN III (Kurokawa et al. 2008) module can drive at least four modules.
- RC modules are organized in a lattice structure.

In Sect. 2, we present the general reshaping problem studied in this paper that is significantly different from the reconfigurable problem studied in Hou and Shen (2010, 2014). In particular, Hou and Shen considered to minimize the number of disconnecting/reconnecting operations in a reconfiguration problem, while we consider to minimize both the number of disconnecting/reconnecting operations and the number of reshaping steps (i.e. the number of rotation operations) in a reshaping problem. In this paper, we show that this general optimal reshaping problem with practical constraints on RC modules is NP-complete. Proving that this optimization problem is NP-complete offers good evidence for its intractability. Accordingly, it is unlikely to be able to find efficient (i.e. polynomial-time) algorithm for solving it. Therefore, it is worth designing polynomial-time approximation algorithms that output a sub-optimal solution. In this paper, we also provide the upper and lower bounds of the global optimum, which can be used to facilitate the evaluation and comparison of the approximation algorithms in the future research.

This paper is organized as follows. Our problem formulation is presented in Sect. 2. In Sect. 3, we prove that this optimization problem is NP-complete. In Sect. 4, the upper and lower bounds of the global optimum are presented. Finally, the concluding remark is presented in Sect. 5.

2 Problem formulation

To precisely define our optimal reconfiguration problem, we present the following terminology.

The 3D shape of a SRM-robot consisting of RC modules can be represented by a connected lattice graph (Fig. 2). A lattice graph $G(V, E)$ is a finite-node induced subgraph of the infinite three-dimensional integer grid. A 3D integer coordinate (x_i, y_i, z_i) is assigned to each node $v_i \in V$, which indicates the center position of a RC module.[1] In $G(V, E)$, two nodes v_i and v_j are connected by an edge $e \in E$ if their Manhattan distance is 1, i.e. $|x_i - x_j| + |y_i - y_j| + |z_i - z_j| = 1$. Given this specified edge connection rule, the 3D shape can be simply represented by the node set V in G.

Denote the set of all RC modules in a SRM-robot as S. The shape of a SRM-robot is changed by rotating RC modules. In a single step of the reshaping process:

- a set M of one or more connected RC modules are specified to be moved, and a RC module χ connected to M is specified to perform the rotation operation;
- M is disconnected from $S \backslash \{M \cup \chi\}$ (several disconnecting operations may be needed);

[1] Without loss of generality, we assume that a RC module occupies a space of unit cube.

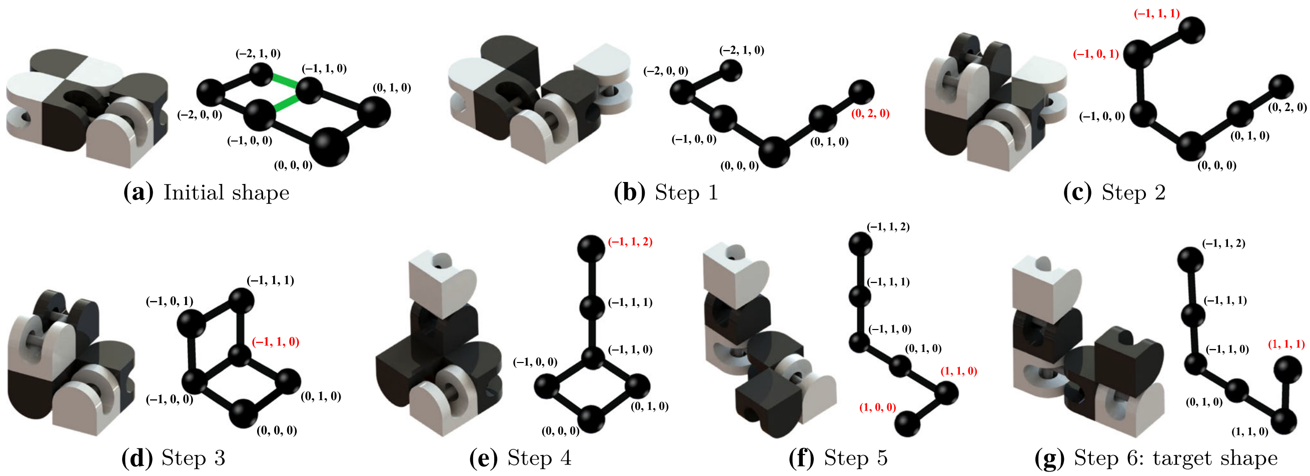


Fig. 2 A reshaping process of a SRM-robot that is made up of three M-TRAN modules. Each M-TRAN module consists of two sub-modules and each sub-module can be regarded as a RC module. The 3D shape—which is made up of six RC modules—is represented by a connected lattice graph $G(V, E)$. Each node $v \in V$ represents a RC module, to which a 3D integer coordinate (x, y, z) is assigned. If two RC modules are touched or connected, there is an edge in $G(V, E)$: green edge for touching (i.e. sharing a face) and black edge for connecting. At each step in the reconfiguration process, the modules whose positions are changed from the previous step are highlighted in red coordinates. From step 2 to step 3, the RC modules at $(-1, 1, 0)$ and $(-1, 0, 0)$ are reconnected at step 3. From step 3 to step 4, the RC modules at $(-1, 0, 0)$ and $(-1, 0, 1)$ are disconnected at step 3. From step 4 to step 5, the RC modules at $(-1, 0, 0)$ and $(-1, 1, 0)$ are disconnected at step 4. Therefore this process needs four disconnecting/reconnecting operations and six steps and then it is a (4, 6) process

– the actuator in χ performs a single orthogonal rotation and drives M to move and change their positions in three-dimensional integer grid;
– M is reconnected to $S \backslash \{M \cup \chi\}$ to stabilize M (several reconnecting operations may be needed).

Definition 1 The cost of a single step in a reshaping process is defined to be the number of disconnecting and reconnecting operations in this step. The cost of a reshaping process is defined to be the total cost of all steps in this reshaping process.

Both the number of disconnecting/reconnecting operations and the number of total steps are two widely used metrics to characterize the efficiency of a SRM-robots' reshaping/reconfiguration process (Pamecha et al. 1997; Hou and Shen 2010, 2014). We characterize the reshaping process by considering both of them:

Definition 2 A reshaping process is called a (p, q) process, if an initial 3D shape can be reshaped into a target 3D shape in q steps with a cost of p.

Figure 2 shows one example of a (4, 6) process.

We define an order on the 2-tuple (p, q), i.e. $(p_1, q_1) < (p_2, q_2)$ if and only if (1) $p_1 < p_2$ or (2) $p_1 = p_2$ and $q_1 < q_2$. In our study, we optimize both p and q by considering the following optimal problem.

Problem 1 (*minimal* (p, q) *problem*) Given an initial shape and a target shape, both having the same number of RC modules, find a reshaping process with the minimal (p, q).

The main result in this paper is summarized below.

Theorem 1 *The minimal* (p, q) *problem is NP-complete.*

We prove Theorem 1 in Sect. 3.

3 NP-completeness of minimal (p, q) problem

3.1 Preliminary

The minimal (p, q) problem is an optimal problem and to prove its NP-completeness, we need to prove that its corresponding decision problem (Problem 2) is NP-complete (Garey and Johnson 1979).

Problem 2 ($C_{(p,q)}$ *problem*) Given an initial shape I and a target shape T (both having the same number of RC modules) and a 2-tuple (p, q), can we take at most q steps with the cost of at most p, such that the shape I can be reshaped into T?

To further prove a decision problem C is NP-complete, we need to show that it is in the NP complexity class and it is also NP-hard, i.e.

– C is in NP: any candidate solution to C can be verified in polynomial time;
– C is NP-hard: any problem in NP is reducible to C in polynomial time.

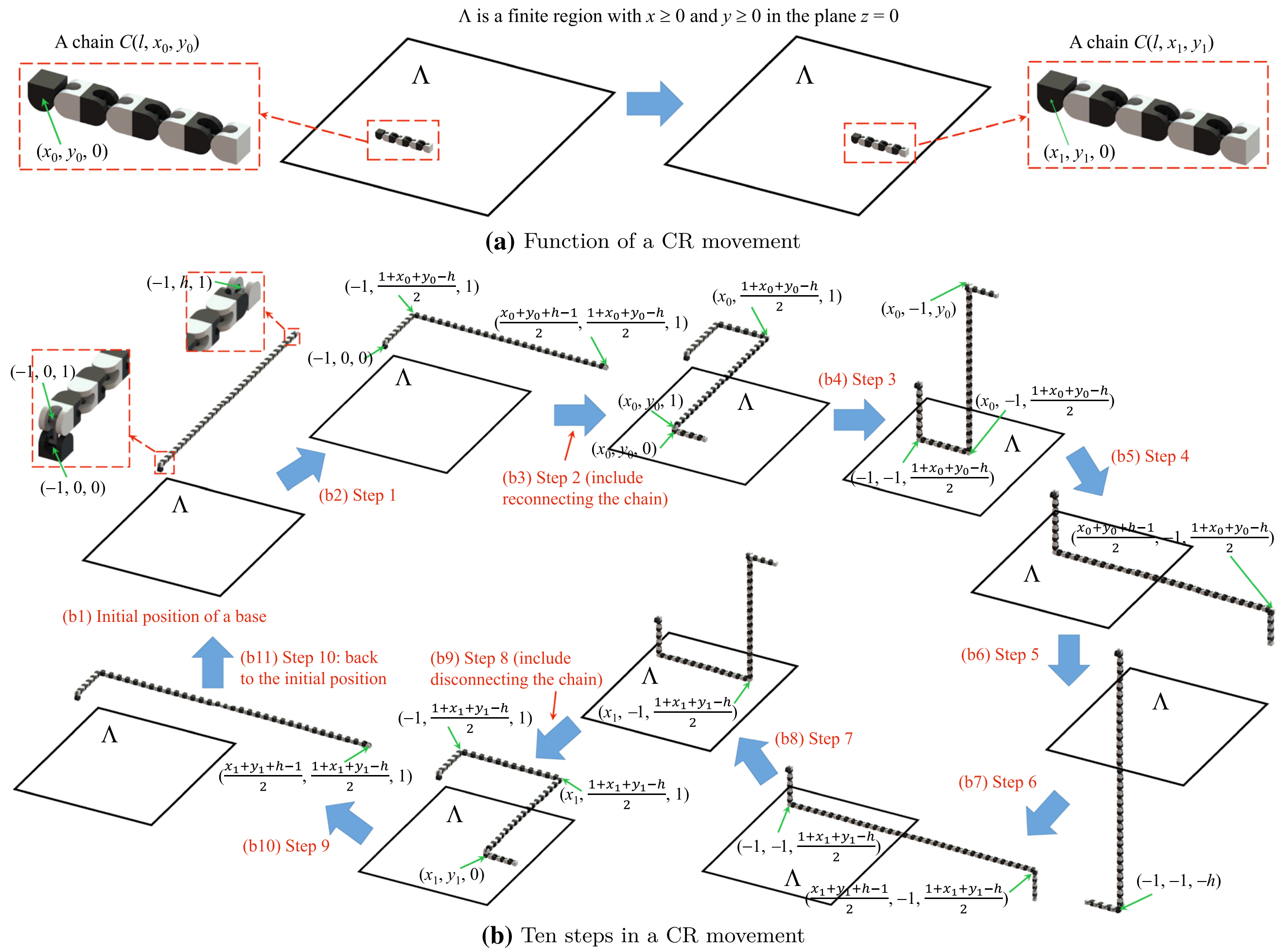


Fig. 3 **a** A CR movement is designed to move a chain of length l from an arbitrary position (x_0, y_0) to another arbitrary position (x_1, y_1) in the area Λ. **b** The detailed ten steps in a reshaping process to realize a CR movement

A problem C_1 is reducible to a problem C_2, denoted by $C_1 \preceq C_2$, if there exists a function f such that

- f maps every instance of C_1 to an instance of C_2, and
- f satisfies that for all instances $x \in C_1$, the reduced problem with instances $f(x) \in C_2$ has the same output as the original problem C_1.

If the reduction function can be computed in polynomial time, C_1 is polynomial-time reducible to C_2, denoted by $C_1 \preceq_P C_2$.

The general strategy for proving a NP problem C is NP-complete is to find a known NP-complete problem C_{NPC} and show $C_{NPC} \preceq_P C$ (Garey and Johnson 1979).

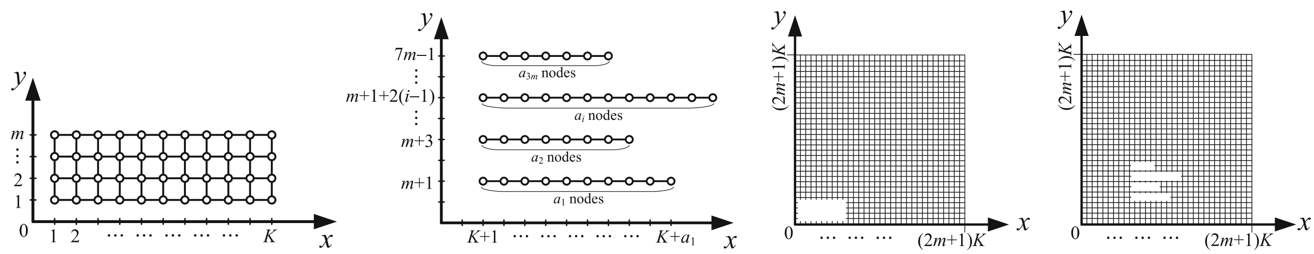
3.2 Overview of the proof

We choose the 3-PARTITION problem (denote as C_{3P}) which is an known NP-complete problem (Garey and Johnson 1979). To ease the presentation of the proof for Theorem 1, the formal definition of the 3-PARTITION problem is summarized in Sect. 3.4.

We denote Problem 2 as $C_{(p,q)}$. Obviously, given any reshaping process with q steps and a cost of p, we can check this process in polynomial time whether the initial shape I can be reshaped into the target shape T. Therefore problem $C_{(p,q)}$ is NP. To prove that $C_{(p,q)}$ is NP complete, our strategy is to design an artificial problem C_a and show that $C_{3P} \preceq C_a \preceq C_{(p,q)}$.

3.3 Problem C_a

We consider a special arrangement of RC modules as illustrated in Fig. 3. Let Λ be a finite area in the plane $z = 0$ satisfying that for any point $p = (x_p, y_p) \in \Lambda$, $x_p \geq 0$ and $y_p \geq 0$. Later, for the problem C_a, Λ will be specified as V_{big_rec} in Eq. (4).



(a) V_{small_rec} **(b)** V_{3m_rows} **(c)** V_I **(d)** V_T

Fig. 4 The construction of node sets V_{small_rec}, V_{3m_rows}, V_I and V_T. All nodes have integer coordinates (x, y) with respect to a 2D coordinate system $\{o, x, y\}$

Definition 3 A chain of length l, denoted as $C(l, x_0, y_0)$, is a set of l connected RC modules in Λ. All the RC modules in $C(l, x_0, y_0)$ have the same fixed coordinates $y = y_0$ and $z = 0$, and the minimal x coordinate in $C(l, x_0, y_0)$ is x_0. i.e. $C(l, x_0, y_0)$ is a chain (1) lies in the $z = 0$ plane, (2) is parallel to the x-axis, and (3) starts at $(x_0, y_0, 0)$ and ends at $(x_0 + l - 1, y_0, 0)$.

Next, we design a special operation called *CR movement* that moves a chain of length l from an arbitrary location (x_0, y_0) to another arbitrary location (x_1, y_1) in Λ.

Definition 4 A base B is a set of connected RC modules, whose geometric pattern is shown in Fig. 3b1 with a sufficiently large number h. Using three steps (Fig. 3b2–b4) and one reconnecting operation, it moves a chain of arbitrary finite length l at an arbitrary location (x_0, y_0) out of the plane $z = 0$ in the direction perpendicular to the plane $z = 0$. Using another five steps (Fig. 3b5–b9) and one disconnecting operation, the base moves the same chain, along the direction perpendicular to the plane $z = 0$, back into the plane $z = 0$ at arbitrary location (x_1, y_1). Additional two steps (Fig. 3b10–b11) is needed to recover the base into the initial pattern. A CR movement is defined to be the set of these continuous rotation (CR) movements in ten steps.

Refer to Fig. 4. Let m and K be two arbitrary positive integers. We construct a special initial shape V_I and a special target shape V_T in the following way. First, both V_I and V_T are set to be planar shapes, i.e. the coordinate $z = 0$ for all the nodes in V_I and V_T. Second, we construct five node sets below, in which all nodes have integer coordinates (x, y) with respect to a 2D coordinate system $\{o, \overrightarrow{x}, \overrightarrow{y}\}$:

– A node set V_{small_rec} defined by (ref. Fig. 4a)

$$V_{small\_rec} = \{(x_i, y_i) : 1 \le x_i \le K, 1 \le y_i \le m\} \quad (1)$$

– A node set V_{3m_rows} which consists of $3m$ subsets (ref. Fig. 4b):

$$V_{3m\_rows} = V_{row\_1} \cup V_{row\_2} \cup \cdots \cup V_{row\_3m} \quad (2)$$

where

$$V_{row\_i} = \{(x_i, y_i) : K + 1 \le x_i \le K + a_i, \quad y_i = m + 1 + 2(i - 1)\}, \quad i = 1, 2, \ldots, 3m \quad (3)$$

– A node set V_{big_rec} defined by

$$V_{big\_rec} = \{(x_i, y_i) : 0 \le x_i, y_i \le (2m + 1)K\} \quad (4)$$

Then we define (ref. Figs. 4c, d)

$$V_I = V_{big\_rec} \backslash V_{small\_rec} \quad (5)$$

and

$$V_T = V_{big\_rec} \backslash V_{3m\_rows} \quad (6)$$

We set $\Lambda = V_{big\_rec}$. The RC modules in Λ are connected in such a special way that when a CR movement moves a chain of length l out of the plane $z = 0$, $l - 2$ disconnecting operations are needed. Then a CR movement consists of ten steps in a reshaping process and contains l disconnecting/reconnecting operations (i.e. plus one reconnecting and one disconnecting operations are needed in steps 2 and 8, respectively; see Fig. 3b3, b9).

We connect the base designed in Fig. 3b1 to both the initial shape V_I and the target shape V_B at the top left corner, denoted as $V_I \cup B$ and $V_T \cup B$, respectively. The special artificial problem C_a we designed is as follows.

Problem 3 (C_a *problem*) Given an initial shape $V_I \cup B$ and a target shape $V_T \cup B$ (both having the same number of RC modules) and a 2-tuple (p, q), can we take at most q steps with the cost of at most p, such that the shape $V_I \cup B$ can be reshaped into $V_T \cup B$?

Since $V_I \cup B$ and $V_T \cup B$ in problem C_a are special cases of the general shapes I and T in the problem $C_{(p,q)}$, it is readily seen that $C_a \preceq C_{(p,q)}$. To show $C_{3P} \preceq C_a \preceq C_{(p,q)}$, we only need to show $C_{3P} \preceq C_a$.

3.4 Proof of $C_{3P} \preceq C_a$

Problem 4 (*3-PARTITION problem C_{3P}*) Given a set A of $3m$ positive integers, denoted by $A = \{a_1, a_2, \ldots, a_{3m}\}$, where $\sum_{a_i \in A} a_i = mK$ and $\frac{K}{4} < a_i < \frac{K}{2}, \forall a_i \in A$. Can A be partitioned into m disjoint subsets $A_1, A_2, \ldots, A_m$, such that $\sum_{a_j \in A_i} a_j = K, i = 1, 2, \ldots, m$?

Since $\frac{K}{4} < a_i < \frac{K}{2}$,we have $|A_i| = 3$, $i = 1, 2, \ldots, m$. Without loss of generality, we can assume $a_i > m$. If this assumption does not hold (i.e. there exists an i, such that $a_i \leq m$), we set $a_i' = a_i + m$, $\forall i$, and $K' = K + 3m$. We have $\sum_{a_i' \in A} a_i' = mK'$, $\sum_{a_j' \in A_i} a_j' = K'$ and $\frac{K'}{4} < \frac{K+4m}{4} < a_i' < \frac{K+2m}{2} < \frac{K'}{2}$, $\forall a_i' \in A$. Therefore, the solution to the 3-PARTITION problem with (a_i, K) is exactly the same to the 3-PARTITION problem with (a_i', K'), where $a_i' > m$.

Given an arbitrary instance $A = \{a_1, a_2, \ldots, a_{3m}\}$ of 3-PARTITION problem C_{3P}, we map it to an instance of problem C_a. The following Lemmas 1 and 2 show that $C_{3P} \preceq C_a$, and thus, complete the proof of Theorem 1.

Lemma 1 (Soundness) *Let $A = \{a_1, a_2, \ldots, a_{3m}\}$ be an arbitrary instance of 3-PARTITION problem. V_I and V_T are initial and target shapes constructed by A using the rules specified in Eqs. (5–6). If 3-PARTITION problem with instance A has a solution, then $V_I \cup B$ can be reshaped into $V_T \cup B$ in a $(mK, 30m)$ process.*

Lemma 2 (Completeness) *$A = \{a_1, a_2, \ldots, a_{3m}\}$ be an arbitrary instance of 3-PARTITION problem. V_I and V_T are initial and target shapes constructed by A using the rules specified in Eqs. (5–6). If $V_I \cup B$ can be reshaped into $V_T \cup B$ in a $(mK, 30m)$ process, then 3-PARTITION problem with instance A has a solution.*

3.4.1 Proof of Lemma 1

If 3-PARTITION problem with instance A has a solution, then the node set V_{small_rec} can be partitioned into m subsets

$$V_i = \{(x_j, y_j) : 1 \leq x_j \leq K, y_j = i\}, \quad i = 1, 2, \ldots, m$$

such that each subset V_i can be filled in by three subsets in V_{3m_rows} [ref. Eq. (2)]:

$$V_i = V_{row\_a_1} \cup V_{row\_a_2} \cup V_{row\_a_3}$$

where $V_{row\_a_j}$, $j = 1, 2, 3$, is determined by $a_j \in A_i$ in the equation $\sum_{a_j \in A_i} a_j = K$ stated in 3-PARTITION problem. By Definition 4, each $V_{row\_a_j}$ can be moved into V_i by a single CR movement. Given that each CR movement takes 10 steps (Fig. 3b), V_I can be transformed into V_T in a $(mK, 30m)$ process.

3.4.2 Proof of Lemma 2

Let ∂V_I and ∂V_T be the outmost boundaries of V_I and V_T, respectively, i.e.

$$\partial V_I = \{(x_i, y_i) \in V_I : x_i = 0 \quad \text{or} \quad x_i = (2m+1)K \\ \text{or} \quad y_i = 0 \text{or } y_i = (2m+1)K\} \tag{7}$$

$$\partial V_T = \{(x_i, y_i) \in V_T : x_i = 0 \quad \text{or} \quad x_i = (2m+1)K \\ \text{or} \quad y_i = 0 \text{or } y_i = (2m+1)K\} \tag{8}$$

We define a special rigid body transformation called *covering transformation* (T_C) of V_I (i.e. including rotation, translation and reflection) such that $\partial T_C(V_I) = \partial V_T$. There are totally eight possible T_C, denoted by $(T_{C1}, T_{C2}, \ldots, T_{C8})$, as shown in Fig. 5.

The subsets V_{small_rec} and V_{3m_rows} that are used to construct V_I and V_T in Eqs. (5–6) satisfy the following result.

Proposition 1 $T_{Ci}(V_{small\_rec}) \cap V_{3m\_rows} = \emptyset$, $i = 1, 2, \ldots 8$.

Proof We only prove the cases with T_{C2} and T_{C3} (ref. Fig. 5). The other cases can be proved similarly.

First, by Eq. (3), the maximal y coordinate of nodes in V_{3m_rows} is

$$y_{\max}(V_{3m\_rows}) = m + 1 + 2(3m - 1) = 7m - 1 \tag{9}$$

Second, the maximal x coordinate of nodes in V_{3m_rows} is

$$x_{\max}(V_{3m\_rows}) = \max_{i \in \{1,2,\ldots,8\}} \{K + a_i\} \\ \leq (m+1)K - (3m-1) \tag{10}$$

In above formulation, we use the fact $a_i \leq mK - (3m - 1)$, which is the result from $\sum_{a_i \in A} a_i = mK$ and A contains $3m$ positive integers in 3-PARTITION problem.

Refer to Fig. 5b. In the case with T_{C2}, the minimal x coordinate in $T_{C2}(V_{small\_rec})$ is

$$(2m+1)K - (K+1) > (m+1)K - (3m-1) \\ \geq x_{\max}(V_{3m\_rows})$$

Therefore $T_{C2}(V_{small\_rec}) \cap V_{3m\_rows} = \emptyset$.

Refer to Fig. 5c. In the case with T_{C3}, the minimal y coordinate in $T_{C3}(V_{small\_rec})$ is

$$(2m+1)K - (K+1) > 7m - 1 = y_{\max}(V_{3m\_rows})$$

In above formulation, we use the fact $K > 3$, which is the result from $|A_i| = 3$, $a_j > m$ and $\sum_{a_j \in A_i} a_j = K$. Therefore $T_{C3}(V_{small\_rec}) \cap V_{3m\_rows} = \emptyset$. □

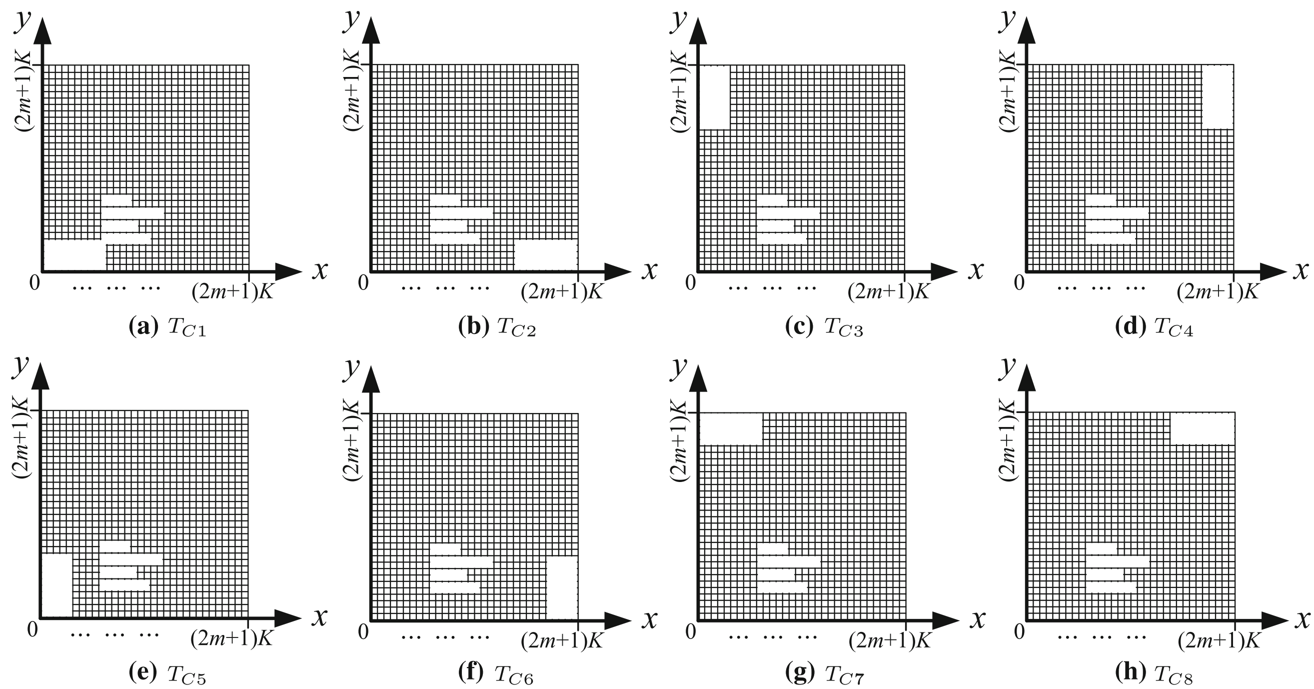


Fig. 5 Eight possible covering transformations (T_C), each of which is a rigid body transformation of V_I (i.e. including rotation, translation and reflection) such that $\partial T_{Ci}(V_I) = \partial V_T$, $i = 1, 2, \ldots, 8$

Second, we define an orthogonal rigid body transformation (ORBT) T_O, which is an orthogonal rotation followed by a translation. To move the modules as few as possible in the reconfiguration process, we apply an ORBT T_O to V_I, such that $T_O(V_I)$ overlap with the nodes in V_T as many as possible.

Proposition 2 *For all possible ORBT T_O, $T_O(V_I)$ can at most overlap with $((2m+1)K+1)^2 - 2mK$ nodes in V_T.*

Proof The proof is in two parts:

- If $\partial T_O(V_I) \neq \partial V_T$, then at least one side in ∂V_T is not overlapped with $\partial T_O(V_I)$. Note that each side has $(2m+1)K+1$ nodes. Then $T_O(V_I)$ and V_T can at most overlap with $((2m+1)K+1)^2 - ((2m+1)K+1) < ((2m+1)K+1)^2 - 2mK$ nodes.
- If $\partial T_O(V_I) = \partial V_T$, then T is one of eight possible T_C. By Lemma 3, $T_{Ci}(V_{small\_rec}) \cap V_{3m\_rows} = \emptyset$, $i = 1, 2, \ldots, 8$. Then all the nodes in $T_{Ci}(V_{small\_rec})$ and V_{3m_rows} are not overlapped. Therefore, $T_O(V_I)$ and V_T overlap with $((2m+1)K+1)^2 - 2mK$ nodes. □

To minimize the cost of the reshaping process, we need to maximize the number of nodes that do not need to be reconfigured in any CR movement, i.e. maximize the set $S \backslash (\bigcup_i M_i)$, where M_i is the set of RC modules reconfigured in ith CR movement. Therefore, we must start at the configuration in which $T_C(V_I)$ can overlap with the maximal number of nodes $((2m+1)K+1)^2 - 2mK$ in V_T. With this configuration, if $T_C(V_I)$ can be transformed into V_T in a $(mK, 30m)$ process, the nodes in $T_C(V_{3m\_rows}) \in T_C(V_I)$ should be transformed into the nodes in $V_{small\_rec \in V_T}$ in $3m$ CR movements.[2] Since the $3m$ subsets in V_{3m_rows} [ref. Eq. (2)] are disconnected with each other and each CR movement can only move a connected subset, each CR movement must move a subset V_{row_i} into $T(V_{small\_rec})$. Since $\forall i$, $a_i > m$, its corresponding subset V_{row_i} must be moved into V_T at the position with the fixed y coordinate, i.e. akin to a row sub-vector in a matrix formed by V_T. Therefore, these $3m$ CR movements correspond to a solution to 3-PARTITION problem with instance A.

4 Bounds of the global solution to minimal (p, q) problem

Showing that the general $C_(p, q)$ problem (Problem 2) is NP-complete offers a good evidence for its intractability, i.e. it is unlikely to find efficient (polynomial-time) algorithms to solve it. Therefore, it is desired to develop polynomial-time approximation algorithms that always output a feasible solution which is close to the global optimum. From this perspective, quantitative bounds of the 2-tuples (p, q) is valuable for evaluating these approximation algorithms. In this

[2] Noting that each CR movement takes 10 steps (Fig. 3b), $30m$ steps are required in the reshaping process.

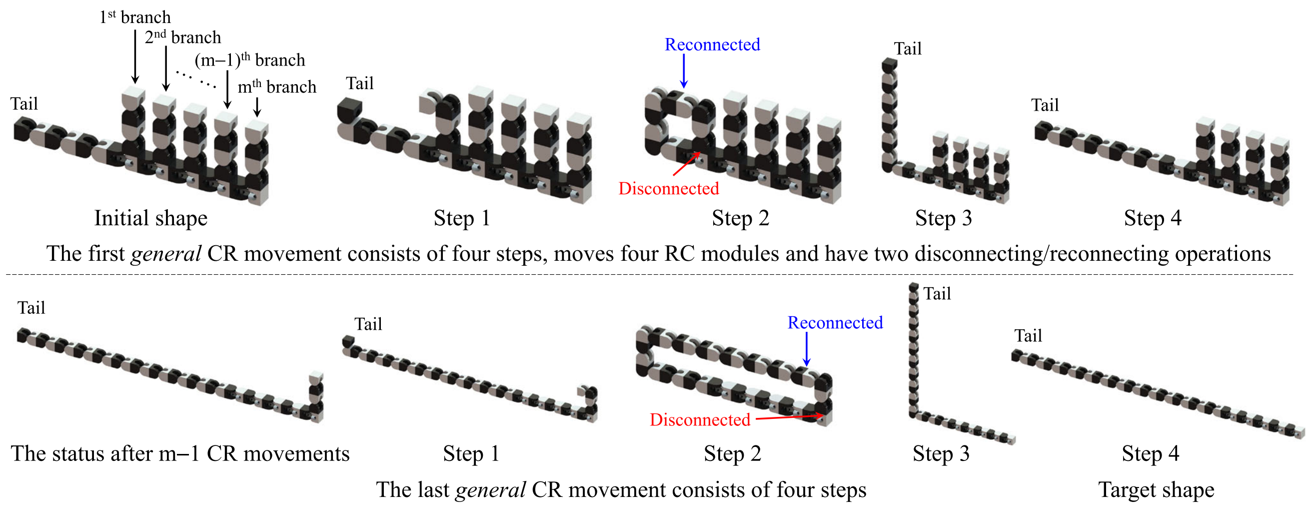


Fig. 6 An example shows that the lower bound $\left(\Omega(\max\{c_i, c_g\}), \Omega(\max\{c_i, c_g\})\right)$ is reachable for the general $C_(p, q)$ problem. In this example, $n = |V_I| = 6m+6$, $n' = |\widetilde{T}_O(V_I) \cap V_T| = 2m+6$, $c_i = m$ and $c_g = 1$. Then $\max\{c_i, c_g\} = m$, which is exactly the solution $(\Omega(m), \Omega(m))$. See Sect. 4.1 for details

section, we establish the both lower and upper bounds for (p, q) in the general $C_(p, q)$ problem.

4.1 Lower bound

Given an arbitrary initial and target shapes V_I and V_T, we apply an ORBT $\widetilde{T}_O$ such that $\widetilde{T}_O(V_I)$ overlaps with the maximal number of nodes in V_T. $\widetilde{T}_O$ can be computed by enumerating all possible ORBTs. Since in $\mathbb{R}^3$ there are only 24 possible orthogonal rotations and a finite number of translation with which $T_O(V_I) \cap V_T \neq \emptyset$, $\widetilde{T}_O$ can be computed in polynomial time. Let

$$n' = |\widetilde{T}_O(V_I) \cap V_T| \tag{11}$$

and

$$n = |V_I| = |V_T| \tag{12}$$

The non-overlapped $n - n'$ nodes in V_I must be moved in the reshaping process. Let c_i and c_g be the number of connected components in $\widetilde{T}_O(V_I) \backslash V_T$ and $V_T \backslash \widetilde{T}_O(V_I)$, respectively. It is straightforward to compute c_i and c_g in linear time by using either breadth-first search or depth-first search. Each set of connected components in either c_i or c_g requires at least one step for transformation in the reshaping process, and each step requires at least one disconnecting/reconnecting operation. Therefore, the lower bound of (p, q) in the general $C_(p, q)$ problem is $\left(\Omega(\max\{c_i, c_g\}), \Omega(\max\{c_i, c_g\})\right)$, where Ω is the notation for an asymptotic lower bound (Cormen et al. 1990):

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n), \quad \forall n \ge n_0\} \tag{13}$$

Below we present an example, showing that the lower bound $\left(\Omega(\max\{c_i, c_g\}), \Omega(\max\{c_i, c_g\})\right)$ is reachable and thus is tight. Refer to Fig. 6. The initial shape V_I is a short straight chain connected with m separated branches. The target shape V_T is a long straight chain. The SRM-robot consists of $3m+3$ M-TRAN modules, which can be regarded as having $6m+6$ RC modules equivalently. We define a *general* CR movement as follows:

- the tail of the SRM-robot (i.e. one end of the chain without branches) lifts up to connect with the nearest branch (at one end of the branch) and then this branch is disconnected with the chain (at the another end of the branch);
- the tail rolls back to be again in straight shape.

Each general CR movement consists of 6 steps,[3] moves 4 connected RC modules and uses 2 disconnecting/reconnecting operations. In the reshaping process of this example, each general CR movement moves one branch and totally m general CR movements are needed. Then the solution (p, q) in this example is $(\Omega(m), \Omega(m))$. Given that $c_i = m$ and $c_g = 1$ in this example, we have $\max\{c_i, c_g\} = m$ and the lower bound $(\Omega(\max\{c_i, c_g\}), \Theta(\max\{c_i, c_g\}))$ is exactly $(\Omega(m), \Omega(m))$.

[3] The four steps shown in Fig. 6 actually take six rotation steps; i.e. each of steps 1 and 3 contains two orthogonal rotations.

4.2 Upper bound

Any polynomial time algorithm that finds a solution to the following reshape planning problem provides an upper bound to (p, q) in the general $C_(p, q)$ problem.

Problem 5 (*reshape planning problem*) Given an initial shape V_I and a target shape V_T, both having the same number n of RC modules, find a reshaping process that transforms V_I to V_T.

Given our practical constraints on regular lattice representation and motion by orthogonal rotations, it is very difficult to design even a heuristic algorithm for solving Problem 5. Here, based on the state-of-the-art work (Sung et al. 2015), we establish an upper bound.

Sung et al. (2015) proposed a heuristic algorithm for M-Block (Romanishin et al. 2015, 2013), which was a kind of RC modules. A pivoting operation was used to move a M-Block with the following conditions:

- a pivoting operation rotates a module about an edge that this module shares with another module;
- a module pivots by the maximum angle possible (either $\frac{\pi}{2}$ or π) until it contacts another module.

Then a pivoting operation can be regarded as a CR movement that takes one step and has one disconnecting and one reconnecting operations. Let n' and n be defined as in Eqs. (11)–(12). Sung et al.' algorithm moves each of $n - n'$ modules in $\widetilde{T}_O(V_I) \backslash (\widetilde{T}_O(V_I) \cap V_T)$ from $\widetilde{T}_O(V_I)$ to V_T, and each movement takes $O(n')$ steps, where O is the notation for an asymptotic upper bound (Cormen et al. 1990):

$$O(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq fg(n) \leq cg(n), \quad \forall n \geq n_0\} \tag{14}$$

Then the solution (p, q) in this algorithm is $(O(n'(n - n')), O(n'(n - n')))$, which is an upper bound of the general $C_(p, q)$ problem.

5 Conclusion

In this paper, we study the SRM-robot consisting of RC modules and formulate its optimal reconfiguration planning problem as a minimal (p, q) problem, by considering both the number of disconnecting/reconnecting operations and the number of reshaping steps (i.e. the number of rotation operations) in a reshaping process. A detailed computational complexity analysis on this problem is presented, showing that this problem is NP-complete. This result offers a good evidence that a poly-nomial-time algorithm is unlikely to exist for this optimal problem. We also provide the lower and upper bounds for the 2-tuples (p, q), which can be used to evaluate the performance of approximation algorithms studied in future research.

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